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Division 2, National Defense Research Committee of the
Office of Scientific Research and Development

MONTHLY REPORT NO. AES-12 (OSRD NO. 5393)

AIR AND EARTH SHOCK

Volume 12. June 25 to July 25, 1945

A Compilation of Informal Reports Submitted in Advance of Formal Reports

Pertinent Service Project
OD-03

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DECLASSIFIED - DOD Directive No. 5200.9, 27 September 1958

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E Bright Wilson Jr

E. Bright Wilson, Jr., Chief
Division 2
Effects of Impact and Explosion

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Preface

This report is the twelfth monthly report of Division 2, NDRC, on Air and Earth Shock, covering the period from June 25 to July 25, 1945. These monthly reports are compilations of informal reports submitted in advance of formal reports. In no case is it to be presumed that the work is complete or that the results reported are other than tentative.

The work described in the report is pertinent to the project designated by the War Department Liaison Officer as OD-03 and was performed under Contract OEMsr-260 with Princeton University.

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July 25, 1945
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Project OD-03

Princeton University
W. Bleakney, Supervisor

REACTIONS OF SIMPLE SYSTEMS UNDER BLAST LOADING

by D. Montgomery and A. H. Taub

Abstract

The differential equation $M\ddot{x} + F(x) = p(t)$ is considered for some simple cases of blast loading. The right-hand side is assumed linear, and $F(x)$ on the one hand is taken as constant and on the other is taken as linear from the origin to the constant and then as remaining constant for larger values of x. It is shown that the situations in the two cases differ moderately. An approximation formula is developed by which certain information in the latter case can be obtained from the former.

1. Introduction

In discussing the behavior of various targets under blast loading it is often possible to reduce the mathematical problem to that of a one-dimensional system governed by the equation

$$M \frac{d^2x}{dt^2} + F(x) = p(t), \tag{1}$$

where x is the displacement of the system, $F(x)$ is the restoring force, M is the equivalent mass of the system, and $p(t)$ is the force [= pressure × area] acting on the system where $p(t)$ is dependent on time.

Equations of this form arise in many problems; for example, if the target is elastic and has various modes of vibration, its response is determined by solving a set of equations of the type of Eq. (1) where $F(x)$ is of the form $k_n x$. Again, this equation is found in the treatment given by Christopherson[1/] in R.C. 349 of the action of brick walls. There it is shown that $F(x)$ may be replaced by a constant.

In the application we have in mind (blast wave) the function $p(t)$ is zero for negative time, has a finite initial value p_o at $t = 0$, decreases to

1/ "A modification of the impulse criterion for blast damage," by D. G. Christopherson, R.C. 349, Sept. 1942 (Confidential).

zero again at time t_0, and becomes negative thereafter, rising to zero at some later time. The problem with which we are mainly concerned may be stated as follows:

What relation must exist between p_0 and t_0 in order that the maximum of the solution of Eq. (1) be a specified quantity, say x_3?

If the solution of this problem is known, then for a target such as a brick wall we can determine the relation between p_0 and t_0 that will just cause the target to fail -- that is, reach a critical displacement with zero velocity. The quantity p_0 is called the peak pressure acting on the target, and t_0 is called the duration of the pressure wave. The area under the pressure-time curve between $t = 0$ and $t = t_0$, called the positive impulse, may be related to p_0 and t_0. The result may then be expressed in terms of the peak pressure and positive impulse just necessary to cause failure. If the relation between peak pressure and impulse acting on the target and the same quantities in the blast wave are known, then for any charge weight a distance can be determined that is the limiting distance at which the target is destroyed. In order to perform the last calculation the dependence of peak pressure and impulse in the blast wave on weight of charge and distance must be known. These quantities must be corrected for reflection, diffraction, and motion of the target in order to obtain the peak pressure and impulse acting on the target.

This paper will be concerned with the determination of the relation between peak pressure and impulse <u>acting on the target</u> for a given maximum displacement for special cases of Eq. (1). The specializations made are as follows:

$$\text{A:} \quad p(t) = p_0\left(1 - \frac{t}{t_0}\right).$$

$$\text{Case I:} \quad F(x) = \text{constant} = P,$$

or

$$\text{Case II:} \quad F(x) = \begin{cases} \dfrac{P}{x_1}\, x, & 0 \leq x \leq x_1; \quad \text{(a)} \\ P, & x \geq x_1. \quad \text{(b)} \end{cases}$$

Case I is a limiting case of case II. If the desired deflection is x_3 and if x_1/x_3 approaches zero, then general existence theorems guarantee that the solutions in case II approach those in case I. However, there is no guarantee that a given value of x_1/x_3, say 0.01 for example, will bring the solutions near each other to an accuracy of about the same size. Actually we find that the solutions can differ to a greater degree than 0.01 in this case, although the difference is not excessive. We exhibit numerical calculations bearing on this point, and we also develop a formula that makes it easy to calculate from the limiting case what the situation is for a given value of x_1/x_3 provided this value is not too large. We consider only cases where $x_3 > x_1$ since in such cases a target will be destroyed when it reaches a deflection x_3 with zero velocity. At the end we also take up a related question whose description we postpone.

2. Solution for case II

We shall treat case I as a special case of case II and proceed first to obtain the solutions in the latter case.

In the interval from 0 to x_1 the solution is as follows:

$$x = \frac{Po}{M\omega^2}\left[\frac{\sin \omega t}{\omega t_o} - \frac{t}{t_o} + 1 - \cos \omega t\right], \tag{2}$$

where $\omega^2 = P/Mx_1$, and hence in this interval

$$\dot{x} = \frac{Po}{M\omega^2 t_o}[\cos \omega t + \omega t_o \sin \omega t - 1]. \tag{3}$$

Let t_1 be the time at which the displacement reaches x_1. Making use of the fact that $\omega^2 = P/Mx_1$, we see that

$$x_1 = \frac{Pox_1}{P}\left[\frac{\sin \omega t_1}{\omega t_o} - \frac{t_1}{t_o} + 1 - \cos \omega t_1\right].$$

Dividing by x_1 and rearranging, we find as the equation determining t_1

$$\omega t_o \cos \omega t_1 + \omega t_1 - \sin \omega t_1 = \omega t_o\left(1 - \frac{P}{Po}\right). \tag{4}$$

We let $\dot{x}_1$ be the value of $\dot{x}$ at t_1 and we denote by I the quantity $\frac{1}{2}p_o t_o$ which is the area under the curve $p(t)$ from 0 to t_o. Then from Eq. (3)

$$\frac{M\dot{x}_1}{I} = \frac{2}{\omega^2 t_o^2}(\omega t_o \sin \omega t_1 + \cos \omega t_1 - 1). \tag{5}$$

When t is greater than t_1 and x is greater than x_1 Eq. (1) becomes

$$M\ddot{x} = p(t) - P = (p_o - P) - \frac{p_o}{t_o} t.$$

Making use of the fact that $x = x_1$ and $\dot{x} = \dot{x}_1$ when $t = t_1$, we find that

$$M\dot{x} = M\dot{x}_1 + \left[p_o\left(1 - \frac{t_1}{t_o}\right) - P\right](t - t_1) - \frac{p_o}{2t_o}(t - t_1)^2 \tag{6}$$

and

$$Mx = Mx_1 + M\dot{x}_1(t - t_1) + \frac{1}{2}\left[p_o\left(1 - \frac{t_1}{t_o}\right) - P\right](t - t_1)^2 - \frac{p_o}{6t_o}(t - t_1)^3. \tag{7}$$

Let t_3 be the value of t at which the solution given by Eq. (7) has its maximum value, and let x_3 be this maximum. When $t = t_3$ the left-hand side of Eq. (6) is zero and we obtain

$$M\dot{x}_1 = \frac{p_o}{2t_o}(t_3 - t_1)^2 - \left[p_o\left(1 - \frac{t_1}{t_o}\right) - P\right](t_3 - t_1).$$

Let $\tau_3 = (t_3 - t_1)/t_o$. Rearrangement gives

$$\frac{M\dot{x}_1}{I} = \tau_3^2 - 2\left[\left(1 - \frac{t_1}{t_o}\right) - \frac{P}{p_o}\right]\tau_3.$$

Solving for τ_3,

$$\tau_3 = \left(1 - \frac{t_1}{t_o} - \frac{P}{p_o}\right) \pm \left[\frac{M\dot{x}_1}{I} + \left(1 - \frac{t_1}{t_o} - \frac{P}{p_o}\right)^2\right]^{1/2}, \tag{8}$$

where we must choose the sign before the square root so as to make τ_3 positive. In developing the approximation formula we consider the case where $1 - t_1/t_o - P/p_o$ is positive.

To find x_3 we substitute this value in Eq. (7):

$$Mx_3 = Mx_1 + t_o\tau_3\left\{M\dot{x}_1 + \frac{1}{2}\left[p_o\left(1 - \frac{t_1}{t_o}\right) - P\right]t_o\tau_3 - \frac{p_o t_o}{6}\tau_3^2\right\}.$$

Replace $M\ddot{x}_1$ by its value from the equation preceding Eq. (8):

$$M\ddot{x}_3 = M\ddot{x}_1 + \frac{p_o t_o^2 \tau_3^2}{6}\left[2\tau_3 - 3\left(1 - \frac{t_1}{t_o} - \frac{P}{p_o}\right)\right]. \tag{8a}$$

Now $p_o = \frac{p_o}{P}\,P = \frac{p_o}{P}\,M\omega^2 x_1$, and hence

$$x_3 = x_1 + \frac{1}{6}\,\frac{p_o}{P}\,\omega^2 t_o^2 x_1 \tau_3^2\left[2\tau_3 - 3\left(1 - \frac{t_1}{t_o} - \frac{P}{p_o}\right)\right]. \tag{9}$$

Instead of plotting p_o against t_o it is more convenient to plot $I/\sqrt{2PM\ddot{x}_3}$ against p_o/P, and we shall next derive a formula expressing the first of these quantities in terms of the second in the limiting case. In this case $t_1 = x_1 = 0$. Here Eq. (9) becomes meaningless because in Eq. (9) we have used the expression P/x_1 for a slope. However, from Eq. (8a), in this case

$$M\ddot{x}_3 = \frac{t_o^2\tau_3^2 p_o}{6}\left[2\tau_3 - 3\left(1 - \frac{P}{p_o}\right)\right],$$

and also in this same case by Eq. (8)

$$\tau_3 = 2\left(1 - \frac{P}{p_o}\right).$$

Hence

$$2PM\ddot{x}_3 = \frac{16PI^2}{3p_o}\left(1 - \frac{P}{p_o}\right)^3 \tag{10}$$

and

$$\frac{I^2}{2PM\ddot{x}_3} = \frac{3}{16}\,\frac{p_o}{P}\,\frac{1}{\left(1 - \frac{P}{p_o}\right)^3}. \tag{11}$$

From Eq. (11) a table of values may be computed relating p_o/P and $I/\sqrt{2PM\ddot{x}_3}$. Table I and the graph of $x_1/x_3 = 0$ in Fig. 1 present these values.

Table I. Values of $I/\sqrt{2PM\ddot{x}_3}$ in the limiting case.

$\frac{p_o}{P}$	$\frac{I}{\sqrt{2PM\ddot{x}_3}}$	$\frac{p_o}{P}$	$\frac{I}{\sqrt{2PM\ddot{x}_3}}$
1.25	5.413	6	1.394
1.5	2.756	7	1.444
2	1.732	8	1.496
3	1.378	9	1.550
4	1.333	10	1.604
5	1.353		

Notice that for some computations in case II it is convenient to use the following relation

$$\frac{I}{\sqrt{2PM\ddot{x}_3}} = \frac{\frac{1}{2}\,\frac{p_o}{P}\,\omega^2 x_1 M t_o}{\sqrt{2M^2\omega^2 x_1 x_3}} = \frac{1}{2\sqrt{2}}\,\frac{p_o}{P}\sqrt{\frac{x_1}{x_3}}\,\omega t_o. \tag{12}$$

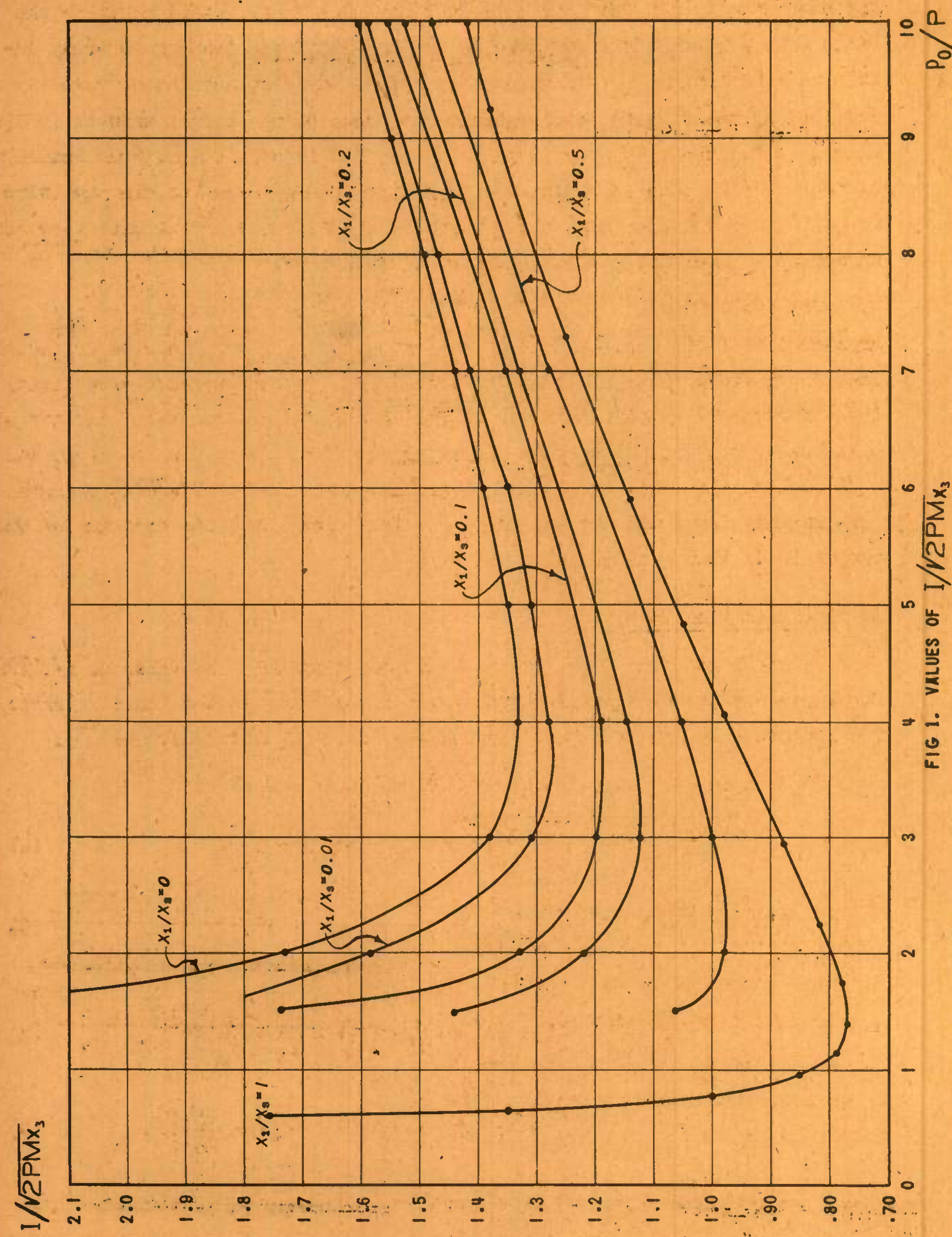

FIG 1. VALUES OF $1/\sqrt{2PMx_3}$

Princeton University Station

Division 2, NDRC

The quantity $I^2/2PMx_3$ has the following physical interpretation; it is the ratio of the kinetic energy given to the target if the loading is truly impulsive (the impulse $\underline{I}$ is communicated before any displacement or velocity is acquired by the target) to the static work done on the target when it is displaced to failure. This ratio would be one for impulsive loading. Actually in the limiting case this ratio is a function of P/p_o and its minimum value is 4/3. Thus the fact that the loading is spread out over a finite time has an appreciable effect on the behavior of the system.

The fact that the value of $I/\sqrt{2PMx_3}$ rises slowly for values of p_o/P greater than four, implies that in this range of p_o/P the "impulse criterion" is approximately true. That is, if the impulse in the pressure wave acting on the structure is greater than and approximately the minimum value the target will break, provided of course p_o/P is greater than four. In the range where p_o/P is less than two, but greater than one, the value of $I/\sqrt{2PMx_3}$ changes very rapidly for small changes in p_o/P. This means that the breaking of the target is following a pressure criterion.

3. Approximation formulas

We shall now find a method to obtain approximately the value of $I/\sqrt{2PMx_3}$ for a given value of x_1/x_3 from the value of $I/\sqrt{2PMx_3}$ in the limiting case. The equations we need for this purpose are Eqs. (4), (5), (8), and (9).

For convenience we collect these formulas in one place

$$\omega t_o \cos \omega t_1 + \omega t_1 - \sin \omega t_1 = \omega t_o\left(1 - \frac{P}{p_o}\right), \tag{4}$$

$$\frac{M\dot{x}_1}{I} = \frac{2}{\omega^2 t_o^2}\left(\omega t_o \sin \omega t_1 + \cos \omega t_1 - 1\right), \tag{5}$$

$$\tau_3 = \left(1 - \frac{t_1}{t_o} - \frac{P}{p_o}\right) + \left[\frac{M\dot{x}_1}{I} + \left(1 - \frac{t_1}{t_o} - \frac{P}{p_o}\right)^2\right]^{1/2}, \tag{8}$$

$$x_3 = x_1 + \frac{x_1}{6}\frac{p_o}{P}\omega^2 t_o^2 \tau_3^2\left[2\tau_3 - 3\left(1 - \frac{t_1}{t_o} - \frac{P}{p_o}\right)\right]. \tag{9}$$

Suppose that P/p_0 is fixed. If we also fix ωt_0, then ωt_1 is determined by Eq. (4). Then $M\dot{x}_1/I$ is given by Eq. (5) and, since $t_1/t_0 = \omega t_1/\omega t_0$, τ_3 is given by Eq. (8) and x_1/x_3 is determined by Eq. (9). Hence for each value of P/p_0 there will be a value of ωt_0 which gives x_1/x_3 a fixed value.

Assuming that x_1/x_3 and P/p_0 are fixed we now estimate what the value of ωt_0 will be when x_1/x_3 is small.

For a first rough estimate assume that $t_1 = 0$ and that $\tau_3 = 2(1 - P/p_0)$ as in the limiting case. Then from Eq. (9)

$$\frac{x_3}{x_1} = 1 + \frac{2}{3}\frac{p_0}{P}\,\omega^2 t_0^2\left(1 - \frac{P}{p_0}\right)^3.$$

We may drop the one as unimportant when x_3/x_1 is large and get as a first estimate $(\omega t_0)_1$ of ωt_0,

$$(\omega t_0)_1^2 = \frac{3\,\frac{x_3}{x_1}\,\frac{P}{p_0}}{2\left(1 - \frac{P}{p_0}\right)^3}. \qquad (13)$$

We now assume that a second approximation $(\omega t_0)_2$ is given by

$$(\omega t_0)_2 = \beta(\omega t_0)_1, \qquad (14)$$

where β is a quantity to be determined.

From Eq. (4) it is seen that a good approximation to ωt_1 is

$$\omega t_1 = \arccos\left(1 - \frac{P}{p_0}\right), \qquad (15)$$

and from Eq. (5) an approximation for $M\dot{x}_1/I$ is

$$\frac{M\dot{x}_1}{I} = \frac{2\sin\omega t_1}{\omega t_0}. \qquad (16)$$

We also make the following estimates for τ_3 and τ_3^2:

$$\tau_3 = 2\left(1 - \frac{t_1}{t_0} - \frac{P}{p_0}\right) + \frac{M\dot{x}_1}{I}\,\frac{1}{2\left(1 - \frac{t_1}{t_0} - \frac{P}{p_0}\right)} \qquad (17)$$

and

$$\tau_3^2\left[2\tau_3 - 3\left(1 - \frac{t_1}{t_0} - \frac{P}{p_0}\right)\right] = 4\left(1 - \frac{P}{p_0}\right)^3 - 12\left(1 - \frac{P}{p_0}\right)^2\frac{t_1}{t_0} + 6\,\frac{M\dot{x}_1}{I}\left(1 - \frac{P}{p_0}\right). \qquad (18)$$

The value $(\omega t_o)_1$ has been chosen so that the desired value of x_3/x_1 is given by

$$\frac{x_3}{x_1} = \frac{2}{3}\frac{P_o}{P}(\omega t_o)_1^2\left(1 - \frac{P}{P_o}\right)^3, \tag{19}$$

but substitution of Eq. (18) in Eq. (9) and use of the second approximation $(\omega t_o)_2 = \beta(\omega t_o)_1$ gives

$$\frac{x_3}{x_1} = 1 + \frac{1}{6}\frac{P_o}{P}\beta^2(\omega t_o)_1^2\left[4\left(1 - \frac{P}{P_o}\right)^3 - 12\left(1 - \frac{P}{P_o}\right)^2\frac{t_1}{t_o} + 6\frac{M\dot{x}_1}{I}\left(1 - \frac{P}{P_o}\right)\right].$$

Equating these two values, again dropping the one as unimportant when x_3/x_1 is large, and dividing, we find

$$4\left(1 - \frac{P}{P_o}\right)^3 = \beta^2\left[4\left(1 - \frac{P}{P_o}\right)^3 - 12\left(1 - \frac{P}{P_o}\right)^2\frac{t_1}{t_o} + 6\frac{M\dot{x}_1}{I}\left(1 - \frac{P}{P_o}\right)\right].$$

Hence

$$\beta^2 = \frac{1}{1+\alpha},$$

where

$$\alpha = \frac{1}{1 - \frac{P}{P_o}}\left[\frac{3/2}{1 - \frac{P}{P_o}}\frac{M\dot{x}_1}{I} - 3\frac{t_1}{t_o}\right].$$

Substituting the relation given by Eq. (16) and remembering that $t_1/t_o = \omega t_1/\omega t$,

$$\alpha = \frac{3}{\left(1 - \frac{P}{P_o}\right)^2}\left[\sin\omega t_1 - \left(1 - \frac{P}{P_o}\right)\omega t_1\right]\frac{1}{(\omega t_o)_2},$$

and then using the relations given in Eqs. (13), (14), and (15),

$$\alpha = \frac{1}{\beta}\frac{3}{\left(1 - \frac{P}{P_o}\right)^2}\left[\sin\left\{\arccos\left(1 - \frac{P}{P_o}\right)\right\} - \left(1 - \frac{P}{P_o}\right)\arccos\left(1 - \frac{P}{P_o}\right)\right]\left[\frac{2\left(1 - \frac{P}{P_o}\right)^3\frac{x_1}{x_3}}{3\frac{P}{P_o}}\right]^{1/2}$$

or

$$\alpha = \frac{\gamma\sqrt{\frac{x_1}{x_3}}}{\beta},$$

where

$$\gamma = 3\left[\frac{2\frac{p_o}{P}}{3\left(1-\frac{P}{p_o}\right)}\right]^{1/2}\left[\sin\left\{\arc\cos\left(1-\frac{P}{p_o}\right)\right\} - \left(1-\frac{P}{p_o}\right)\arc\cos\left(1-\frac{P}{p_o}\right)\right].$$

Hence

$$\beta^2 = \frac{1}{1+\alpha} = \frac{1}{1+\dfrac{\gamma\sqrt{\frac{x_1}{x_3}}}{\beta}},$$

$$\beta = -\frac{\gamma\sqrt{\frac{x_1}{x_3}}}{2} + \sqrt{1+\frac{\gamma^2 x_1/x_3}{4}},$$

$$\beta \cong 1 - \frac{\gamma}{2}\sqrt{\frac{x_1}{x_3}} + \frac{\gamma^2}{8}\frac{x_1}{x_3}. \qquad (20)$$

Table II. Values of γ and β.

$\frac{p_o}{P}$	γ	$\frac{\gamma}{2}$	$\frac{\gamma^2}{8}$	β for $\frac{x_1}{x_3} = 0.01$
1.25	4.323	2.162	2.337	0.760
1.5	2.767	1.384	0.958	.852
2	1.677	0.838	.351	.920
3	0.959	.480	.115	.953
4	.688	.344	.0592	.966
5	.522	.261	.0341	.974
6	.425	.212	.0225	.979
7	.359	.180	.0162	.982
8	.311	.156	.0122	.984
9	.274	.137	.0094	.986
10	.245	.122	.0074	.988

Thus we have achieved our purpose — to find the approximate value of β. A table of values of $\gamma/2$ and $\gamma^2/8$ as well as the values of β when $x_1/x_3 = 0.01$ is given in Table II.

If we replace ωt_o in Eq. (12) by $(\omega t_o)_1$ we see that we obtain the value of $I/\sqrt{2PMx_3}$ for the limiting case. Hence the factor β is also the factor which when multiplied by $I/\sqrt{2PMx_3}$ in the limiting case yields the value (approximately) for any value of x_1/x_3.

Table III shows $I/\sqrt{2PMx_3}$ as computed in certain cases and as given by the approximation formula derived above. It can be seen that the approximation formula is quite accurate in these cases.

Table III. Comparison of exact and approximate values of $I/\sqrt{2PMx}$

$\frac{p_o}{P}$	$\frac{x_1}{x_3}$	Exact Value of $\frac{I}{\sqrt{2PMx_3}}$	Value Given b Approximation Formula
1.5	0.0205	2.280	2.265
2.0	.0424	1.457	1.458
2.0	.0124	1.577	1.578
3.0	.0150	1.300	1.299
3.0	.0069	1.324	1.324
4.0	.0223	1.269	1.266
4.0	.0129	1.283	1.281
7.0	.0091	1.420	1.420
10.0	.0220	1.575	1.575
10.0	.0125	1.583	1.582

Hence, to get a good estimate of $I/\sqrt{2PMx_3}$ for a given small value of x_1/x_3, compute β from Eq. (20), using in many cases the values of $\gamma/2$ and $\gamma^2/8$ from Table II. Then multiply β and the limiting value of $I/\sqrt{2PMx_3}$ from Table I. This approximation formula is quite accurate when x_1/x_3 is small, and is fairly accurate for values of x_1/x_3 as large as 0.3 or 0.4. Figure 1 gives the values of $I/\sqrt{2PMx_3}$ for various values of x_1/x_3 between 0 and 1. The case $x_1/x_3 = 1$ may be handled as follows. In this case F(x) is a straight line and[2/]

$$x_3 = \frac{2p_o}{M\omega^2}\left[1 - \frac{\arctan \omega t_o}{\omega t_o}\right];$$

using the fact that $\omega^2 = \frac{P}{Mx_3}$, we obtain

$$\frac{P}{p_o} = 2\left[1 - \frac{\arctan \omega t_o}{\omega t_o}\right].$$

We find also in this case, by Eq. (12),

$$\frac{I}{\sqrt{2PMx_3}} = \frac{1}{2\sqrt{2}}\frac{p_o}{P}\,\omega t_o.$$

The graphs for this case and for the cases where $x_1/x_3 = 0.0$, 0.01, 0.1, 0.2, and 0.5 are shown in Fig. 1.

The curves of Fig. 1 all have vertical asymptotes on the left. To find them proceed as follows. Looking at the Eqs. (4), (5), (8), (9), and (12),

2/ The following equation is derived by solving the differential equation for x(t). Determine the time at which the maximum is obtained from the equation $\dot{x}(t_1) = 0$. Substitute this value of t_1 in the equation $x(t_1) = x_3$. See R.C. 6, "The design of buildings against air attack (Part 2)," March 1939. Restricted.

let ωt_0 approach infinity. Then τ_3 approaches zero. Let $\frac{p_0}{P}$, $\frac{1}{2} \leq \frac{p_0}{P} \leq 1$, be fixed, and attempt to find the corresponding value of x_1/x_3. Since $1 - t_1/t_0 - P/p_0$ is negative, the approximation for τ_3 given by Eq. (17) is to be replaced by the expression obtained by the choice of signs in Eq. (8) which makes τ_3 positive. This gives

$$\tau_3 = \frac{-M\dot{x}}{I} \frac{1}{2(1 - t_1/t_0 - P/p_0)}.$$

Since $t_1/t_0 = \omega t_1/\omega t_0$ is negligible compared to $1 - P/p_0$, we obtain

$$\tau_3 = \frac{\sin \omega t_1}{\omega t_0 (P/p_0 - 1)}.$$

Substituting in Eq. (19) and replacing $\sin^2 \omega t_1$ by $2P/p_0 - (P/p_0)^2$ we obtain

$$\frac{x_3}{x_1} = 1 + \frac{1}{6} \frac{p_0}{P} \frac{2P/p_0 - (P/p_0)^2}{(P/p_0 - 1)^2} [\tau_3 - 3(1 - P/p_0)].$$

Making use of the fact that τ_3 is small compared to $3(1 - P/p_0)$ we write

$$\frac{x_3}{x_1} = 1 + \frac{1}{6} \frac{2 - P/p_0}{(P/p_0 - 1)^2} 3(P/p_0 - 1)$$

$$= 1 + \frac{2 - P/p_0}{P/p_0 - 1}$$

and

$$\frac{x_1}{x_3} = \frac{2P/p_0 - 2}{P/p_0}.$$

This may be written

$$\frac{p_0}{P} = 1 - \frac{1}{2} \frac{x_1}{x_3}.$$

Hence this is the location of the vertical asymptotes for the curve associated with x_1/x_3. When $x_1/x_3 = 0$ the asymptote is at 1, and as x_1/x_3 increases to 1 the position of the asymptote shifts linearly to $\frac{1}{2}$.

As we have seen, the curves all have a vertical asymptote given as above. After this they drop rather soon to a minimum and then rise gradually. The position of the minimum varies from about 1.5 for $x_1/x_3 = 1$ to 4 for $x_1/x_3 = 0$.

4. Comparison of case II to an elastic system

Returning to the differential equation Eq. (1), we discuss next a problem which arises in connection with $F(x)$ as given in case II and as given in still another case called case III.

The function $F(x) = kx$ where $k = P/x_1$ is the value given in (a) of case II Thus the curve of case III is merely a continuation of the straight line with which the curve in case II begins. Suppose that the desired maximum deflection in case II is x_3 and that the desired maximum deflection in case III is x_3'. The area A_2 under the curve II from 0 to x_3 is

$$A_2 = kx_1x_3 - \frac{kx_1^2}{2}.$$

The area under the curve III from 0 to x_3' is

$$A_3 = \frac{kx_3'^2}{2}.$$

Under some conditions it is reasonable to suppose that if $A_2 = A_3$, then the p_0 and t_0 which produce a maximum deflection x_3 in case II will produce a maximum deflection x_3' in case III. This conjecture will be examined below, and it will be shown that it is not always true.

For $A_2 = A_3$, the following must hold

$$x_3'^2 = 2x_1x_3 - x_1^2,$$

or when x_1 is small,

$$x_3'^2 = 2x_1x_3. \tag{21}$$

Let x_{III} be the maximum deflection in case III. Then

$$x_{III} = \frac{2p_0}{M\omega^2}\left[1 - \frac{\text{arc tan } \omega t_0}{\omega t_0}\right]$$

$$= 2\,\frac{p_0}{P}\,x_1\left[1 - \frac{\text{arc tan } \omega t_0}{\omega t_0}\right].$$

For small values of x_1 this is approximated fairly well by

$$x_{III} = 2\,\frac{p_0}{P}\,x_1.$$

In case II assume that x_1/x_3 is so small that the maximum deflection x_{II} is approximately equal to what it would be in the limiting case,

$$x_{II} = \frac{1}{M}\frac{2}{3}\left(1 - \frac{P}{p_o}\right)^3 p_o t_o^2.$$

According to Eq. (21) we wish to compare the quantities

$$4\left(\frac{p_o}{P}\right)^2 x_1^2 \quad \text{and} \quad \frac{4}{3M}\left(1 - \frac{P}{p_o}\right)^3 p_o t_o^2 x_1.$$

It is clear that in general these two quantities do not approximate each other, and as a further check it is easy to choose special values of the constants which show a substantial discrepancy between the two quantities.

As a numerical example suppose that $M = 1$, $t_o = 1$, $\frac{P}{p_o} = \frac{1}{2}$ and $\frac{x_{II}}{x_1} = 100$.

Then

$$x_{II} = \frac{p_o}{12},$$

$$x_{III} = 4x_1,$$

and

$$\frac{x_{II}}{x_1} = 100 = \frac{p_o}{12x_1},$$

$$p_o = 1200x_1.$$

We wish to compare the quantities

$$16x_1^2 \quad \text{and} \quad \frac{p_o x_1}{6},$$

or

$$16x_1 \quad \text{and} \quad \frac{p_o}{6},$$

or

$$16x_1 \quad \text{and} \quad 200x_1.$$

These quantities differ by a factor of more than 12. For this numerical case practically all of the action takes place while the right-hand side of the differential equation is positive.

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The differential equation $M\ddot{x} + F(x) = p(t)$ is considered for some simple cases of blast loading. The right-hand side is assumed linear, and F(x) on the one hand is taken as constant and on the other is taken as linear from the origin to the constant and then as remaining constant for larger values of x. It is shown that the situation in the two cases differ moderately. An approximation formula is developed by which certain information in the lstter case can be obtained from the former.

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